A repairable queueing model with two-phase service, start-up times and retrial customers

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Abstract

A repairable queueing model with a two-phase service in succession, provided by a single server, is investigated. Customers arrive in a single ordinary queue and after the completion of the first phase service, either proceed to the second phase or joins a retrial box from where they retry, after a random amount of time and independently of the other customers in orbit, to find a position for service in the second phase. Moreover, the server is subject to breakdowns and repairs in both phases, while a start-up time is needed in order to start serving a retrial customer. When the server becomes idle, he departs for a single vacation of an arbitrarily distribution length. The arrival process is assumed to be Poisson and all service and repair times are arbitrarily distributed. For such a system the stability conditions and steady state analysis are investigated. Numerical results are finally obtained and used to investigate system performance.

Keywords: Poisson arrivals, Two-phase service, Retrial queue, Breakdowns, Repairs, Start-up time, Vacation.

1 Introduction

The main characteristics of the queueing model analysed in this paper are (i) the retrial customers (jobs), (ii) the server breakdowns and repairs, (iii) the two-phase service and (iv) the start-up (system preparation) times.

Queueing systems with repeated attempts (retrials) are characterized by the feature that an arriving customer who finds the server unavailable, leaves the system, joins a pool of unsatisfied customers, the so-called retrial box, and repeats his demand for service after a random amount of time. Retrial queues have been widely used to model many problems in telephone switching systems, telecommunications networks and computer units. For a complete survey on this topic we refer Artalejo [3], Kulkarni and Liang [17], and the books of Falin and Templeton [13], and Artalejo and Gomez-Corral [5].

In most of the queueing literature, the server is assumed to be reliable and always available to customers. However in practice, we often meet cases where the server may breakdown and has to be repaired. In queueing literature, there have been several works taking into account both retrial phenomenon and server breakbowns with repairs. As related works we mention the papers of Aissani [1], Aissani and Artalejo [2], Kulkarni and Choi [16], Wang et al. [23].

The assumption of a two-phase service provided by a single server has been proved useful to analyse many practical situations arrising in packet transmissions, multimedia communications, central processors etc.. Such kind of systems have been discussed for the first time by Krishna and Lee [15] and Doshi [12], and more recently have been generalized to include models with vacations, N-policy etc. (see [6], [9], [14]).

Wang [22], considered a two-phase queueing model with the assumptions of breakdowns and repairs, in which he assumed that the second optional service follows an exponential distribution. Kumar, Vijayakumar, Arivudainambi [18], Artalejo and Choudhury [4], and Choudhury [7] are the first who imposed the concept of retrial customers in the two phase models. The common feature of the above papers is that there are no server breakdowns, no ordinary queue and all waiting customers join the retrial box. Choudhury and Deka [8], generalize the works of, Wang [22], and Artalejo and Choudhury [4] by considering an M/G/1 retrial queue with second optional service channel which is subject to server breakdowns and repair. Wang and Li [24], consider a similar model, where only the first retrial customer can retry for service after an arbitrarily distributed time period.

Recently Dimitriou and Langaris [11], considered a two-phase model where all arriving customers are queued up in a single ordinary queue. After the completion of the first phase service the customer either proceeds to the second phase or joins the retrial box from where he retries, after a random amount of time, to find the server available, and to complete his second phase of service.

In this work we generalize the model of Dimitriou and Langaris [11], allowing server breakdowns and repairs in both phases of service, while in addition, the server needs a start-up (system preparation) time in order to start serving a retrial customer in the second phase of service. Our system can be used to model any situation with two stages of service where in the first stage a control and a separation of the serviced units, according to some quality standards or some measure of importance, must been taking place. If a unit satisfies these quality standards then it proceeds immediately to the second phase of service while if the quality of the unit is poor then it is removed from the system and repeats its attempt to receive a special second service later when the server is free from high quality units. Moreover the machine (server), is naturally subject to breakdowns and repairs while a special preparation of the machine is needed to start serving the low quality unit. Such a situation often arises in packet transmissions, manufacturing systems, central processors, multimedia communications, etc.. It is clear that the concepts of breakdowns, repairs, and the start-up period for the retrial customer, make our model more realistic compared with models analysed in the above mentioned works.

The article is organized as follows. A full description of the model is given in Section 2. Some very useful for the analysis, results on the customer completion time and server busy period are given in Section 3. In Section 4 the conditions for statistical equilibrium are investigated. The generating functions of the steady state probabilities are obtained in Section 5 and used to give, in Section 6, some important measures of the system performance. Finally in Section 7, numerical results are obtained and used to compare system performance under various changes of the parameters.

2 The Model

Consider a queueing system consisting of two phases of service and a single server, who follows the customer in service when he passes from the first phase to the second. Customers arrive to the system according to a *Poisson* process with parameter λ, and are placed in a single queue waiting to be served. When a customer finishes his service in the first phase, he either goes to the second with probability $1-p$, or he joins, with probability p, a retrial box from where he retries, independently to the other customers in the box, after an exponential time parameter α, to find a position for service in the second phase. In case the customer chooses to join the retrial box the server starts immediately to serve in the first phase the next customer (if any) in queue. Let us denote by P_1, the customers who are waiting in the ordinary queue or are in any phase of service but without joining the retrial box and by P_2, those who joined the retrial box and are still there or are now in their second phase of service.

To start serving a P_2 (retrial) customer, in the second phase, the server needs a start-up period S, which is arbitrarily distributed with distribution function (D.F.) $S(x)$, probability density function (p.d.f.) $s(x)$, finite mean value $\bar{s}$ and second moment about zero $\bar{s}^{(2)}$. If a P_1 customer arrives during S, this start-up period is interrupted, the server start serving the P_1 customer in the first phase and the P_2 customer returns to the retrial box.

Every time the server becomes idle (no customers waiting in the ordinary queue) he departs for a single vacation B_0 which length is arbitrarily distributed with D.F. $B_0(x)$, p.d.f. $b_0(x)$, finite mean value $\bar{b}_0$ and second moment about zero $\bar{b}_0^{(2)}$.

The server is subject to breakdowns and repairs in both phases of service. Thus the server's lifetime is assumed to be exponential with parameter ν_{ij} when he serves a P_i customer in the jth phase while the repairing time in the jth phase is assumed to be arbitrarily distributed with D.F. $R_j(x)$, p.d.f. $r_j(x)$, finite mean value $\bar{r}_j$, and second moment about zero $\bar{r}_j^{(2)}$. Moreover if a breakdown occurs in first phase (a P_1 customer in service) the customer just being served goes back to the head of the queue, waiting the server to be repaired and to start, from scrats, his first phase service again. On the other hand if a breakdown occurs during the second phase service of a P_1 customer, the interrupted customer remains in the service zone and start service, from scrats, upon repair completion. If a breakdown occurs when the server serves a P_2 customer (in the second phase of course), the P_2 customer returns to the retrial box and the server, upon repair completion, starts serving a P_1 customer (if any) in the first phase or he remains free and departs for a vacation.

The service times in both phases are assumed to be arbitrarily distributed with D.F. $B_{ij}(x)$, p.d.f. $b_{ij}(x)$, finite mean value $\bar{b}_{ij}$ and second moment about zero $\bar{b}_{ij}^{(2)}$ for the P_i customer in the jth phase respectively $i, j = 1, 2$, ($B_{21}(x)$, $b_{21}(x)$, $\bar{b}_{21}, \bar{b}_{21}^{(2)}$ do not exist). Finally all random variables defined above are assumed to be independent.

3 Preliminary Results

We agree from here on to denote in general by $a^*(s)$, the Laplace-Stieltges Transform (LST) of any function $a(t)$. Let us define now by F the time interval from the epoch a P_1 customer starts his service in the second phase until this period be succesfully completed. Let also $N_1(F)$, be the number of new P_1 customers that arrive during F. Note here, that during this period no new P_2 customers join the retrial box. Define finally

$$f_i(t)dt = P(t < F \leq t+dt, N_1(F) = i), \qquad f^*(z_1, s) = \int_0^\infty e^{-st} \sum_{i=0}^\infty f_i(t) z_1^i dt$$

and denote $p_i(t) = e^{-\lambda t} \frac{(\lambda t)^i}{i!}$. Then it is clear that

$$f_i(t) = e^{-\nu_{12} t} p_i(t) b_{12}(t)$$
$$+\nu_{12} e^{-\nu_{12} t} \sum_{m=0}^{i} p_m(t)(1 - B_{12}(t)) * \sum_{n=0}^{i-m} p_n(t) r_2(t) * f_{i-m-n}(t)$$

where $*$ means convolution. Thus after manipulations

$$f^*(z_1, s) = \frac{\beta_{12}^*(s + \lambda + \nu_{12} - \lambda z_1)}{1 - \nu_{12} \frac{1-\beta_{12}^*(s+\lambda+\nu_{12}-\lambda z_1)}{s+\lambda+\nu_{12}-\lambda z_1} r_2^*(s + \lambda - \lambda z_1)}.$$

Denote by S_1 the time interval from the epoch at which a P_1 customer starts his service in first phase until the epoch the server is ready for a "new service". We have to point out here that in case of a P_1 customer, a "new service" starts, either after the end of the two-phase service procedure, or after the repair caused by a breakdown in first phase, or because the customer joins the retrial box. Let also S_2 be the time interval from the epoch at which a P_2 customer finds a position for service in the second phase until the epoch the server is ready for a "new service". In case of a P_2 customer a "new service" starts, either after the completion of the second phase service, or after a possible arrival of a P_1 customer during the start-up period, or after the repair caused by a breakdown in second phase. Denote also by $N_i(S_j)$ the number of new P_i customers that arrive during S_j. If we define, for $j = 1, 2$

$$a_j(k_1, k_2, t)dt = \quad P(t < S_j \leq t + dt, \ N_i(S_j) = k_i, \ i = 1, 2),$$
$$a_j^*(z_1, z_2, s) = \quad \int_0^\infty e^{-st} \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty a_j(k_1, k_2, t) z_1^{k_1} z_2^{k_2} dt,$$

then it is easy to see that

$$
\begin{aligned}
a_1^*(z_1,z_2,s) = \quad & \nu_{11} z_1 \tfrac{1-\beta_{11}^*(s+\lambda+\nu_{11}-\lambda z_1)}{s+\lambda+\nu_{11}-\lambda z_1} r_1^*(s+\lambda-\lambda z_1) \\
& +\beta_{11}^*(s+\lambda+\nu_{11}-\lambda z_1)[pz_2+(1-p)f^*(z_1,s)], \\
a_2^*(z_1,z_2,s) = \quad & \tfrac{\lambda z_1 z_2(1-s^*(\lambda+s))}{\lambda+s} + s^*(\lambda+s)[\beta_{22}^*(s+\lambda+\nu_{22}-\lambda z_1) \\
& +\nu_{22} z_2 \tfrac{1-\beta_{22}^*(s+\lambda+\nu_{22}-\lambda z_1)}{s+\lambda+\nu_{22}-\lambda z_1} r_2^*(s+\lambda-\lambda z_1)].
\end{aligned}
\tag{1}
$$

Let us denote

$$
\begin{aligned}
\rho_1 = \tfrac{d}{dz_1} a_1^*(z_1,1,0)|_{z_1=1} = \quad & 1-\beta_{11}^*(\nu_{11})+\lambda[(1-\beta_{11}^*(\nu_{11}))(\bar{r}_1+\tfrac{1}{\nu_{11}}) \\
& +\tfrac{(1-p)\beta_{11}^*(\nu_{11})(1-\beta_{12}^*(\nu_{12}))}{\beta_{12}^*(\nu_{12})}(\bar{r}_2+\tfrac{1}{\nu_{12}})].
\end{aligned}
\tag{2}
$$

To proceed further we need the following Lemma the proof of which is a simple application of the well known theorem of Takacs [21].

Lemma 1 *For (i)* $|z_2|<1$, $\mathrm{Re}(s)\geq 0$, *or (ii)* $|z_2|\leq 1$, $\mathrm{Re}(s)>0$, *or (iii)* $|z_2|\leq 1$, $\mathrm{Re}(s)\geq 0$ *and* $\rho_1>1$, *the relation*

$$z_1 - a_1^*(z_1,z_2,s), \tag{3}$$

has one and only one root, $z_1 = x(s,z_2)$, *say, inside the region* $|z_1|<1$. *Specifically for* $s=0$ *and* $z_2=1$, $x(0,1)$ *is the smallest positive real root of (3) with* $x(0,1)<1$ *if* $\rho_1>1$ *and* $x(0,1)=1$ *for* $\rho_1\leq 1$.

Let us denote now by $\mathcal{B}^{(i)}$ the duration of a busy period of P_1 customers which starts with $i=1,2,\ldots$ P_1 customers, and let $\mathcal{N}(\mathcal{B}^{(i)})$ be the number of new P_2 customers joining the retrial box during $\mathcal{B}^{(i)}$. Define

$$g_m^{(i)}(t)dt = \Pr[t<\mathcal{B}^{(i)}\leq t+dt,\ \mathcal{N}(\mathcal{B}^{(i)})=m].$$

Then it is known from Langaris and Katsaros [19] that

$$g^{*(i)}(s,z_2) \equiv \int_0^\infty e^{-st}\sum_{m=0}^\infty g_m^{(i)}(t) z_2^m dt = x^i(s,z_2),$$

where $x(s,z_2)$ is defined in Lemma 1 above.

Let now V, be the random interval from the epoch the server departs for a single vacation until the epoch he is for the first time idle. Let also $N(V)$ be the number of the new P_2 customers joining the retrial box during V and define

$$
\begin{aligned}
v_m(t)dt = \quad & P(t<V\leq t+dt,\ N(V)=m), \\
v^*(s,z_2) = \quad & \textstyle\int_0^\infty e^{-st}\sum_{m=0}^\infty v_m(t) z_2^m dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
& v_0(t) = p_0(t)b_0(t) + \textstyle\sum_{i=1}^\infty p_i(t)b_0(t) * g_0^{(i)}(t) * v_0(t), \\
& v_m(t) = \textstyle\sum_{i=1}^\infty p_i(t)b_0(t) * \sum_{k=0}^m g_k^{(i)}(t) * v_{m-k}(t),
\end{aligned}
\tag{4}
$$

and so after some algebra

$$v^*(s,z_2) = \frac{\beta_0^*(s+\lambda)}{1+\beta_0^*(s+\lambda)-\beta_0^*(s+\lambda-\lambda x(s,z_2))}. \tag{5}$$

In a similar way, denote by C the random interval from the epoch a P_2 (retrial) customer finds a position for service (in the second phase of course) until the epoch the server departs for the single vacation, and let $N(C)$ be the number of the new customers joining the retrial box during C. If

$$c_m(t)dt = P(t < C \le t+dt,\ N(C)=m), \quad c^*(s,z_2) = \int_0^\infty e^{-st}\sum_{m=0}^\infty c_m(t)z_2^m dt$$

then by writing for $c_m(t)$ a similar expression as in (4) we obtain after manipulations

$$c^*(s,z_2) = a_2^*(x(s,z_2),z_2,s). \tag{6}$$

Now we are ready to define the concepts of the Generalized Completion time and Generalized busy period. **Generalized completion time**, W_2 say, of a P_2 (retrial) customer is the time elapsed from the epoch this customer succeed to find a position for service until the epoch the server is idle for the first time, while **generalized busy period**, W_1 say, is the time interval from the epoch a P_1 customer arrives in an idle system until the epoch the server is idle again . If now we denote by $N(W_2)$, $N(W_1)$ the number of new retrial customers joining the retrial box in W_2, W_1, respectively, and define

$$\begin{aligned} w_m^{(i)}(t)dt &= P(t < W_i \le t+dt,\ N(W_i)=m), \\ w_i^*(s,z_2) &= \textstyle\int_0^\infty e^{-st}\sum_{m=0}^\infty w_m^{(i)}(t)z_2^m dt, \end{aligned} \qquad i=1,2,$$

then it is clear that

$$w_2^*(s,z_2) = c^*(s,z_2)v^*(s,z_2), \qquad w_1^*(s,z_2) = x(s,z_2)v^*(s,z_2), \tag{7}$$

and so

$$\begin{aligned} w_2^*(s,z_2) &= \tfrac{c^*(s,z_2)\beta_0^*(s+\lambda)}{1+\beta_0^*(s+\lambda)-\beta_0^*(s+\lambda-\lambda x(s,z_2))}, \\ w_1^*(s,z_2) &= \tfrac{x(s,z_2)\beta_0^*(s+\lambda)}{1+\beta_0^*(s+\lambda)-\beta_0^*(s+\lambda-\lambda x(s,z_2))}. \end{aligned} \tag{8}$$

Differentiating the obtained relations with respect to z_2 at the point $(z_2 = 1, s=0)$ we arrive easily at

$$\begin{aligned} \tfrac{d}{dz_2}x(0,z_2)|_{z_2=1} &= \tfrac{p\beta_{11}^*(\nu_{11})}{1-\rho_1}, \\ \tfrac{d}{dz_2}c^*(0,z_2)|_{z_2=1} &= \tfrac{\rho_2}{1-\rho_1}, \\ \tfrac{d}{dz_2}v^*(0,z_2)|_{z_2=1} &= \tfrac{\rho_0}{1-\rho_1}. \end{aligned}$$

where

$$\begin{aligned}\rho_0 = & \frac{\lambda p \beta_{11}^*(\nu_{11})\bar{b}_0}{\beta_0^*(\lambda)}, \\ \rho_2 = & (1-\rho_1)(1-s^*(\lambda)\beta_{22}^*(\nu_{22})) \\ & +p\beta_{11}^*(\nu_{11})[1-s^*(\lambda)+\lambda s^*(\lambda)(1-\beta_{22}^*(\nu_{22}))(\bar{r}_2+\frac{1}{\nu_{22}})],\end{aligned} \tag{9}$$

and so

$$\begin{aligned}E(N(W_2)) = \frac{d}{dz_2}w_2^*(0,z_2)|_{z_2=1} = \frac{\rho_0+\rho_2}{1-\rho_1}, \\ E(N(W_1)) = \frac{d}{dz_2}w_1^*(0,z_2)|_{z_2=1} = \frac{p\beta_{11}^*(\nu_{11})+\rho_0}{1-\rho_1}.\end{aligned} \tag{10}$$

Moreover, by differentiating relations (8) with respect to s at the point ($z_2 = 1, s = 0$) we obtain the mean duration of W_2, W_1, respectively, as

$$\begin{aligned}E(W_2) = & \frac{\beta_{11}^*(\nu_{11})}{\lambda(1-\rho_1)}[\frac{\lambda\bar{b}_0}{\beta_0^*(\lambda)}+1-s^*(\lambda) \\ & +\lambda s^*(\lambda)(1-\beta_{22}^*(\nu_{22}))(\bar{r}_2+\frac{1}{\nu_{22}})], \\ E(W_1) = & \frac{1}{1-\rho_1}[\frac{\beta_{11}^*(\nu_{11})\bar{b}_0}{\beta_0^*(\lambda)}+(1-\beta_{11}^*(\nu_{11}))(\bar{r}_1+\frac{1}{\nu_{11}}) \\ & +\frac{(1-p)\beta_{11}^*(\nu_{11})(1-\beta_{12}^*(\nu_{12}))}{\beta_{12}^*(\nu_{12})}(\bar{r}_2+\frac{1}{\nu_{12}})].\end{aligned} \tag{11}$$

Let us denote (see relations (2), (9))

$$\rho \equiv \rho_0+\rho_1+\rho_2.$$

We are now ready to state the following theorem.

Theorem 2 *For (i)* $\mathrm{Re}(s) > 0$, *or (ii)* $\mathrm{Re}(s) \geq 0$ *and* $\rho > 1$, *the equation*

$$z_2 - w_2^*(s,z_2) = 0, \tag{12}$$

has one and only one root, $z_2 = \phi(s)$ *say, inside the region* $|z_2| < 1$. *Specifically for* $s = 0$, $\phi(0)$ *is the smallest positive real root of (12) with* $\phi(0) < 1$ *if* $\rho > 1$ *and* $\phi(0) = 1$ *for* $\rho \leq 1$.

Proof: It is clear that $w_2^*(s,z_2)$ is LST of a probability generating function (see (8)). Thus for the closed contour $|z_2| = 1$ and under the assumption (i) we have always (on $|z_2| = 1$)

$$|w_2^*(s,z_2)| \leq w_2^*(Re(s),1) < w_2^*(0,1) = 1 \equiv |z_2|,$$

while for $Re(s) \geq 0$, we need to consider the closed contour $|z_2| = 1-\epsilon$ ($\epsilon > 0$ a small number) in which case

$$|w_2^*(s,z_2)| \leq w_2^*(Re(s),1-\epsilon) < 1-\epsilon \equiv |z_2|, \tag{13}$$

only if in addition

$$\frac{d}{d\epsilon}w_2^*(0,1-\epsilon)\,|_{\epsilon=0} = -\frac{\rho_0+\rho_2}{1-\rho_1} < \frac{d}{d\epsilon}(1-\epsilon)\,|_{\epsilon=0} = -1,$$

or we need $\rho > 1$ for the relation (13) to hold. A final reference to Rouche's theorem completes the first part of the proof.

Moreover for $s = 0$ the convex function $w_2^*(0, z_2)$ is a monotonically increasing function of z_2, for $0 \le z_2 \le 1$, taking the values $w_2^*(0,0) < 1$ and $w_2^*(0,1) = 1$ and so $0 < \phi(0) < 1$ if $\rho > 1$, while for $\rho \le 1$, $\phi(0)$ becomes equal to 1 and this completes the proof. $\square$

4 Stability Conditions

Let $N_1(t)$, $N_2(t)$ be the number of P_1, P_2, customers in the ordinary queue (not in service) and in the retrial box respectively at time t and denote by

$$\xi_t = \begin{cases} 0 & \textit{server on vacation at } t, \\ s & \textit{server on start up at } t, \\ (i,j) & \textit{server busy on } j\text{th } \textit{phase with } P_i \textit{ customer at } t, \\ (r,i,j) & \textit{server under repair from breakdown on } j\text{th } \textit{phase} \\ & \textit{during service of } P_i \textit{ customer at } t, \\ id & \textit{server idle at } t. \end{cases}$$

Consider also the time instants

$$T_0 = 0 < T_1 < T_2 < ...,$$

where T_i is the epoch at which the server becomes idle for the ith time, and let $N_{2i} = N_2(T_i + 0)$, $i = 0, 1, 2, ...$, i.e. N_{2i} denote the number of customers in the retrial box just after T_i . It is clear that the stochastic process $\{N_{2i} : i = 0, 1, 2, ...\}$ is an irreducible and aperiodic Markov chain. The following theorem gives the condition under which this Markov chain becomes positive recurrent.

Theorem 3 *For $\rho < 1$ the Markov chain $\{N_{2i} : i = 0, 1, 2, ...\}$ is positive recurrent.*

Proof: To prove the theorem, we will use the following criterion (see Pakes [20]):

An irreducible and aperiodic Markov chain $(Y_n ; n \ge 0)$, with state space the nonnegative integers, is positive recurrent if $|\delta_k| < \infty$ for all $k = 0, 1, 2, \ldots$ and $\limsup_{k \to \infty} \delta_k < 0$, where $\delta_k = E[Y_{n+1} - Y_n \mid Y_n = k]$.

For the Markov chain of our model, let

$$h_{k,m}(t)dt = \Pr[t < T_{n+1} - T_n \le t + dt,\ N_{2n+1} - N_{2n} = m |\ N_{2n} = k].$$

Then it is easy to see that for $m = 0, 1, 2, ...$

$$h_{k,m}(t) = \lambda e^{-(\lambda + k\alpha)t} * w_m^{(1)}(t) + kae^{-(\lambda + ka)t} * w_{m+1}^{(2)}(t),$$

while for $m = -1$

$$h_{k,-1}(t) = \begin{array}{l} k\alpha e^{-(\lambda+k\alpha)t} * p_0(t)s(t) * e^{-\nu_{22}t}p_0(t)b_{22}(t) * v_0(t) \\ +k\alpha e^{-(\lambda+k\alpha)t} * p_0(t)s(t) * e^{-\nu_{22}t}\sum_{i=1}^{\infty} p_i(t)b_{22}(t) * g_0^{(i)}(t) * v_0(t), \end{array}$$

and so

$$\int_0^{\infty} e^{-st} \sum_{m=-1}^{\infty} h_{k,m}(t)z^m dt = \frac{\lambda w_1^*(s,z) + \frac{k\alpha}{z} w_2^*(s,z)}{s+\lambda+k\alpha}. \tag{14}$$

Differentiating (14) with respect to z at the point $(z = 1, s = 0)$ we arrive at

$$\delta_k = \frac{\lambda E(N(W_1)) + k\alpha[E(N(W_2)) - 1]}{\lambda + k\alpha}, \qquad k = 0, 1, \ldots,$$

where $E(N(W_1))$, $E(N(W_2))$ have been found in (10).

Thus for $\rho < 1$ we realize that $|\delta_k|$ is finite for all k and also $\limsup_{k\to\infty}\delta_k = E(N(W_2))-1 = \frac{\rho_0+\rho_2}{1-\rho_1} - 1 < 0$ (for $\rho < 1$) and the criterion is satisfied. □

For a stochastic process $(Y(t)\ ;\ t \geq 0)$ we will say that it is stable, if its limiting probabilities as $t \to \infty$ exist and form a distribution. Consider now the stochastic process

$$\mathbf{Z} = \{(N_1(t),\ N_2(t),\ \xi_t) :\ 0 \leq t < \infty\},$$

where $N_i(t)$, ξ_t have been defined above. Then

Theorem 4 *For $\rho < 1$ the process $\mathbf{Z}$ is stable.*

Proof: Consider the quantity

$$m_k = E(T_1 |\ N_{20} = k).$$

Differentiating (14) with respect to s (at $z = 1$, $s = 0$) we obtain

$$m_k = \frac{\lambda E(W_1) + k\alpha E(W_2) + 1}{\lambda + k\alpha},$$

and if q_k $k = 0, 1, 2, \ldots$, are the steady state probabilities of the positive recurrent (for $\rho < 1$) Markov chain $\{N_{2i} :\ i = 0, 1, 2, \ldots\}$ then

$$\mathbf{q} \cdot \mathbf{m} = \sum_{k=0}^{\infty} q_k m_k = E(W_2) + \{1 + \lambda[E(W_1) - E(W_2)]\} \sum_{k=0}^{\infty} \frac{q_k}{\lambda + k\alpha}. \tag{15}$$

Now it is clear that there is always a finite integer k^* such that

$$\frac{1}{\lambda + (k^* - 1)\alpha} > 1 > \frac{1}{\lambda + k^*\alpha},$$

and so

$$\sum_{k=0}^{\infty}\frac{q_k}{\lambda+k\alpha} = \sum_{k=0}^{k^*-1}\frac{q_k}{\lambda+k\alpha}+\sum_{k=k^*}^{\infty}\frac{q_k}{\lambda+k\alpha} < \sum_{k=0}^{k^*-1}\frac{q_k}{\lambda+k\alpha} + \sum_{k=k^*}^{\infty} q_k = \sum_{k=0}^{k^*-1}\frac{q_k}{\lambda+k\alpha} + (1-\sum_{k=0}^{k^*-1} q_k) < \infty,$$

and so from (15) using (11) we understand that $\mathbf{q}\cdot\mathbf{m} < \infty$.

Consider finally the irreducible aperiodic and positive recurrent Markov Renewal Process $\{N,\ T\} = \{(N_{2n},\ T_n) :\ n = 0,1,2,...\}$. It is easy to see that the stochastic process $\mathbf{Z}$ is a Semi-Regenerative Process with imbedded Markov Renewal Process $\{N,\ T\}$ and as (for $\rho < 1$) $\mathbf{q}\cdot\mathbf{m} < \infty$ it is clear that $\mathbf{Z}$ is, for $\rho < 1$, stable (Cinlar [10], Theorem 6.12, p. 347). □

5 Steady State Probabilities

Let us assume that $\rho < 1$ and so a state of statistical equilibrium exists for our model. Let also $N_i = \lim_{t\to\infty} N_i(t)$, $i = 1,2$, $\xi = \lim_{t\to\infty}\xi_t$. Define finally for $i,j = 1,2$

$$\begin{aligned}
q(k_2) &= P(N_1 = 0,\ N_2 = k_2,\ \xi = id),\\
p_s(k_2,x)dx &= P(N_1 = 0,\ N_2 = k_2,\ \xi = s,\ x < \bar{S} \le x+dx),\\
p_0(k_1,k_2,x)dx &= P(N_1 = k_1,\ N_2 = k_2,\ \xi = 0,\ x < \bar{B}_0 \le x+dx),\\
p_{ij}(k_1,k_2,x)dx &= P(N_1 = k_1,\ N_2 = k_2,\ \xi = (i,j),\ x < \bar{B}_{ij} \le x+dx),\\
p_{rij}(k_1,k_2,x)dx &= P(N_1 = k_1,\ N_2 = k_2,\ \xi = (r,i,j),\ x < \bar{R}_j \le x+dx),
\end{aligned} \tag{16}$$

where $\bar{X}$ the elapsed duration of any random variable X. If finally

$$\begin{aligned}
Q(z_2) &= \textstyle\sum_{k_2\ge 0} q(k_2)z_2^{k_2},\\
P_s(z_1,z_2,x) &= \textstyle\sum_{k_2\ge 0} p_s(k_2,x)z_2^{k_2},\\
P_0(z_1,z_2,x) &= \textstyle\sum_{k_1\ge 0}\sum_{k_2\ge 0} p_0(k_1,k_2,x)z_1^{k_1}z_2^{k_2},\\
P_{ij}(z_1,z_2,x) &= \textstyle\sum_{k_1\ge 0}\sum_{k_2\ge 0} p_{ij}(k_1,k_2,x)z_1^{k_1}z_2^{k_2},\\
P_{rij}(z_1,z_2,x) &= \textstyle\sum_{k_1\ge 0}\sum_{k_2\ge 0} p_{rij}(k_1,k_2,x)z_1^{k_1}z_2^{k_2},
\end{aligned}$$

then by connecting as usual the probabilities (16) to each other we arrive easily, for $x > 0$, at

$$\begin{aligned}
P_s(z_2,x) &= P_s(z_2,0)(1-S(x))\exp[-\lambda x],\\
P_0(z_1,z_2,x) &= P_0(z_1,z_2,0)(1-B_0(x))\exp[-(\lambda-\lambda z_1)x],\\
P_{ij}(z_1,z_2,x) &= P_{ij}(z_1,z_2,0)(1-B_{ij}(x))\exp[-(\lambda+\nu_{ij}-\lambda z_1)x],\\
P_{rij}(z_1,z_2,x) &= P_{rij}(z_1,z_2,0)(1-R_j(x))\exp[-(\lambda-\lambda z_1)x],
\end{aligned} \tag{17}$$

and

$$\alpha z_2\frac{d}{dz_2}Q(z_2) + \lambda Q(z_2) = P_0(0,z_2,0)\beta_0^*(\lambda). \tag{18}$$

In similar way we obtain for the boundary conditions

$$\begin{aligned}
P_s(z_2,0) = &\ \alpha\tfrac{d}{dz_2}Q(z_2),\\
P_0(0,z_2,0) = &\ pz_2P_{11}(0,z_2,0)\beta_{11}^*(\lambda+\nu_{11}) + P_{r22}(0,z_2,0)r_2^*(\lambda)\\
&+P_{22}(0,z_2,0)\beta_{22}^*(\lambda+\nu_{22}) + P_{12}(0,z_2,0)\beta_{12}^*(\lambda+\nu_{12}),\\
P_{22}(0,z_2,0) = &\ \alpha\tfrac{d}{dz_2}Q(z_2)s^*(\lambda),\\
P_{r11}(z_1,z_2,0) = &\ \nu_{11}z_1\tfrac{1-\beta_{11}^*(\lambda+\nu_{11}-\lambda z_1)}{\lambda+\nu_{11}-\lambda z_1}P_{11}(z_1,z_2,0),\\
P_{r12}(z_1,z_2,0) = &\ \nu_{12}\tfrac{1-\beta_{12}^*(\lambda+\nu_{12}-\lambda z_1)}{\lambda+\nu_{12}-\lambda z_1}P_{12}(z_1,z_2,0),\\
P_{r22}(z_1,z_2,0) = &\ \nu_{22}z_2\tfrac{1-\beta_{22}^*(\lambda+\nu_{22}-\lambda z_1)}{\lambda+\nu_{22}-\lambda z_1}P_{22}(0,z_2,0),
\end{aligned} \tag{19}$$

and

$$P_{12}(z_1,z_2,0) = \frac{(1-p)\beta_{11}^*(\lambda+\nu_{11}-\lambda z_1)P_{11}(z_1,z_2,0)}{1-\nu_{12}\frac{1-\beta_{12}^*(\lambda+\nu_{12}-\lambda z_1)}{\lambda+\nu_{12}-\lambda z_1}r_2^*(\lambda-\lambda z_1)}, \tag{20}$$

while

$$P_{11}(z_1,z_2,0) = \frac{\lambda z_1Q(z_2)+\alpha\frac{d}{dz_2}Q(z_2)a_2^*(z_1,z_2,0)-P_0(0,z_2,0)[1+\beta_0^*(\lambda)-\beta_0^*(\lambda-\lambda z_1)]}{z_1-a_1^*(z_1,z_2,0)}, \tag{21}$$

Replacing now in the numerator of (21) the zero (in $|z_1|<1$) $x(z_2)\equiv x(0,z_2)$ of the denominator we obtain

$$P_0(0,z_2,0) = \frac{\lambda x(z_2)Q(z_2)+\alpha\frac{d}{dz_2}Q(z_2)c^*(0,z_2)}{1+\beta_0^*(\lambda)-\beta_0^*(\lambda-\lambda x(z_2))}. \tag{22}$$

Substituting (22) into (18) we arrive easily at

$$\alpha(z_2-w_2^*(0,z_2))\frac{d}{dz_2}Q(z_2)+\lambda(1-w_1^*(0,z_2))Q(z_2)=0. \tag{23}$$

Let now

$$\omega(z_2)=\frac{1-w_1^*(0,z_2)}{z_2-w_2^*(0,z_2)},$$

then for $\rho<1$ the quantity $z_2-w_2^*(0,z_2)$ never becomes zero in $|z_2|<1$ (Theorem 2) and also

$$\lim_{z_2\to1}\omega(z_2) = -\frac{p\beta_{11}^*(\nu_{11})+\rho_0}{1-\rho}<\infty.$$

Thus $\omega(z_2)$ is an analytic function in $|z_2|<1$ and a continuous one on the boundary and so for any $|z_2|\le1$ we can solve equation (23) and obtain

$$Q(z_2)=Q(1)\exp\{-\frac{\lambda}{\alpha}\int_{z_2}^{1}\frac{1-w_1^*(0,u)}{w_2^*(0,u)-u}du\}. \tag{24}$$

Replacing back $Q(z_2)$ into the generating functions defined above, and demanding the total probabilities to sum to unity we arrive at

$$Q(1) = \frac{1-\rho}{(1+\frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)})\beta_{11}^*(\nu_{11})\beta_{22}^*(\nu_{22})s^*(\lambda)}, \tag{25}$$

and so the generating functions of the steady state probabilities are completely known.

The following theorem shows that the condition $\rho < 1$ is also necessary for a stable system.

Theorem 5 *If the stochastic process* $\mathbf{Z}$ *is stable then* $\rho < 1$.

Proof: Suppose that $\mathbf{Z}$ is stable and $\rho > 1$. Then from Theorem 2 the equation $z_2 - w_2^*(0, z_2) = 0$ has a strictly less than one root ($\phi(0) < 1$) and so $\lambda(1 - w_1^*(0, \phi(0))) \neq 0$. By putting now $\phi(0)$ instead of z_2 in (23) we obtain

$$\lambda(1 - w_1^*(0, \phi(0)))Q(\phi(0)) = 0,$$

and so $Q(\phi(0)) = \sum q(j)\phi^j(0) = 0$ with $0 < \phi(0) < 1$. Thus $q(j) = 0 \ \forall j$ and also from the generating functions in (17)-(22) it is clear that all probabilities become zero. This of course contradicts the hypothesis that the system is stable.

Suppose finally that $\mathbf{Z}$ is stable and $\rho = 1$. Differentiating (23) with respect to z_2 (at $z_2 = 1$) we arrive (for $\rho = 1$) at

$$\frac{d}{dz_2}\lambda(1 - w_1^*(0, z_2))|_{z_2=1}Q(1) = -\lambda E(N(W_1))Q(1) = 0,$$

and so $Q(1) = \sum q(j) = 0$ and this again contradicts the hypothesis that the system is stable.

6 Performance Measures

In the sequel we will use formulas for the generating functions obtained above, to derive expressions for the system performance. Thus by putting $z_1 = z_2 = 1$

into relations (17)-(22) we obtain easily

$$
\begin{aligned}
P[idle] = \quad & Q(1) = \frac{1-\rho}{(1+\frac{\lambda b_0}{\beta_0^*(\lambda)})\beta_{11}^*(\nu_{11})\beta_{22}^*(\nu_{22})s^*(\lambda)}, \\
P[server\ busy\ in\ 1^{st}\ phase] = \quad & P_{11}(1,1) = \frac{\lambda(1-\beta_{11}^*(\nu_{11}))}{\nu_{11}\beta_{11}^*(\nu_{11})}, \\
P[server\ busy\ in\ 2^{nd}\ phase] = \quad & P_{12}(1,1) + P_{22}(1,1) \\
= \quad & \frac{\lambda(1-p)(1-\beta_{12}^*(\nu_{12}))}{\nu_{12}\beta_{12}^*(\nu_{12})} + \frac{\lambda p(1-\beta_{22}^*(\nu_{22}))}{\nu_{22}\beta_{22}^*(\nu_{22})}, \\
P[vacation] = \quad & P_0(1,1) = \frac{\lambda\bar{b}_0[p\beta_{11}^*(\nu_{11})+(1-\rho)/(1+\frac{\lambda\bar{b}_0}{\beta_0^*(\lambda)})]}{\beta_0^*(\lambda)\beta_{11}^*(\nu_{11})\beta_{22}^*(\nu_{22})s^*(\lambda)}, \\
P[repair\ in\ 1^{st}\ phase] = \quad & P_{r11}(1,1) = \frac{\lambda(1-\beta_{11}^*(\nu_{11}))}{\beta_{11}^*(\nu_{11})}\bar{r}_1, \\
P[repair\ in\ 2^{nd}\ phase] = \quad & P_{r12}(1,1) + P_{r22}(1,1) \\
= \quad & [\frac{\lambda(1-p)(1-\beta_{12}^*(\nu_{12}))}{\beta_{12}^*(\nu_{12})} + \frac{\lambda p(1-\beta_{22}^*(\nu_{22}))}{\beta_{22}^*(\nu_{22})}]\bar{r}_2, \\
P[start\ up] = \quad & P_s(1) = \frac{p(1-s^*(\lambda))}{\beta_{22}^*(\nu_{22})s^*(\lambda)}.
\end{aligned}
$$

In order to obtain now the mean number of customers in the ordinary queue (excluding service) and in the retrial box, we have to differentiate relations (17)-(22) with respect to z_1 and z_2 respectively at the point $(z_1, z_2) = (1,1)$. Let us denote $\dot{a}(t)$ the first order derivative of any function $a(t)$. Then after manipulations

$$
\begin{aligned}
&E(N_1, \xi = (1,1)) = \frac{\lambda^2}{\nu_{11}\beta_{11}^*(\nu_{11})}[\frac{(D+\bar{D})(1-\beta_{11}^*(\nu_{11}))}{2(1-\rho_1)} + \dot{\beta}_{11}^*(\nu_{11}) + \frac{1-\beta_{11}^*(\nu_{11})}{\nu_{11}}], \\
&E(N_1, \xi = (1,2)) = \frac{\lambda^2(1-p)}{\nu_{12}\beta_{12}^*(\nu_{12})}\{(1-\beta_{12}^*(\nu_{12}))[\frac{(D+\bar{D})}{2(1-\rho_1)} - \frac{\dot{\beta}_{11}^*(\nu_{11})}{\beta_{11}^*(\nu_{11})} \\
&\qquad + \frac{\dot{\beta}_{12}^*(\nu_{12})+(1-\beta_{12}^*(\nu_{12}))(\bar{r}_2+\frac{1}{\nu_{12}})}{\beta_{12}^*(\nu_{12})}] + \dot{\beta}_{12}^*(\nu_{12}) + \frac{1-\beta_{12}^*(\nu_{12})}{\nu_{12}}\}, \\
&E(N_1, \xi = (r,1,1)) = \frac{\lambda}{\beta_{11}^*(\nu_{11})}\{\bar{r}_1[1-\beta_{11}^*(\nu_{11}) + \lambda(\frac{(D+\bar{D})(1-\beta_{11}^*(\nu_{11}))}{2(1-\rho_1)} \\
&\qquad + \dot{\beta}_{11}^*(\nu_{11}) + \frac{1-\beta_{11}^*(\nu_{11})}{\nu_{11}})] + \frac{\lambda\bar{r}_1^{(2)}}{2}(1-\beta_{11}^*(\nu_{11}))\}, \\
&E(N_1, \xi = (r,1,2)) = \frac{\lambda^2(1-p)}{\beta_{12}^*(\nu_{12})}\{\bar{r}_2[(1-\beta_{12}^*(\nu_{12}))(\frac{(D+\bar{D})}{2(1-\rho_1)} - \frac{\dot{\beta}_{11}^*(\nu_{11})}{\beta_{11}^*(\nu_{11})} \\
&+ \frac{\dot{\beta}_{12}^*(\nu_{12})+(1-\beta_{12}^*(\nu_{12}))(\bar{r}_2+\frac{1}{\nu_{12}})}{\beta_{12}^*(\nu_{12})}) + \dot{\beta}_{12}^*(\nu_{12}) + \frac{1-\beta_{12}^*(\nu_{12})}{\nu_{12}}] + \frac{\bar{r}_2^{(2)}}{2}(1-\beta_{12}^*(\nu_{12}))\} \\
&E(N_1, \xi = (2,2)) = \frac{\lambda^2 p}{\nu_{22}\beta_{22}^*(\nu_{22})}(\dot{\beta}_{22}^*(\nu_{22}) + \frac{1-\beta_{22}^*(\nu_{22})}{\nu_{22}}), \\
&E(N_1, \xi = (r,2,2)) = \frac{\lambda^2 p}{\beta_{22}^*(\nu_{22})}[\bar{r}_2(\dot{\beta}_{22}^*(\nu_{22}) + \frac{1-\beta_{22}^*(\nu_{22})}{\nu_{22}}) + \frac{\bar{r}_2^{(2)}}{2}(1-\beta_{22}^*(\nu_{22}))], \\
&E(N_1, \xi = 0) = \frac{\lambda^2\bar{b}_0^{(2)}}{2\beta_0^*(\lambda)\beta_{11}^*(\nu_{11})\beta_{22}^*(\nu_{22})s^*(\lambda)}(p\beta_{11}^*(\nu_{11}) + \frac{1-\rho}{1+\frac{\lambda\bar{b}_0}{\beta_0^*(\lambda)}}),
\end{aligned}
\tag{26}
$$

and

$$\begin{aligned}
E(N_2, \xi = (1,1)) = &\ \tfrac{1-\beta_{11}^*(\nu_{11})}{\nu_{11}} L, \\
E(N_2, \xi = (1,2)) = &\ \tfrac{(1-p)(1-\beta_{12}^*(\nu_{12}))\beta_{11}^*(\nu_{11})}{\nu_{12}\beta_{12}^*(\nu_{12})} L, \\
E(N_2, \xi = (r,1,1)) = &\ \bar{r}_1(1-\beta_{11}^*(\nu_{11}))L, \\
E(N_2, \xi = (r,1,2)) = &\ \bar{r}_2 \tfrac{(1-p)(1-\beta_{12}^*(\nu_{12}))\beta_{11}^*(\nu_{11})}{\beta_{12}^*(\nu_{12})} L, \\
E(N_2, \xi = s) = &\ \tfrac{1-s^*(\lambda)}{\lambda} M, \\
E(N_2, \xi = (2,2)) = &\ \tfrac{1-\beta_{22}^*(\nu_{22})}{\nu_{22}} s^*(\lambda) M, \\
E(N_2, \xi = (r,2,2)) = &\ \bar{r}_2 s^*(\lambda)(1-\beta_{22}^*(\nu_{22}))[\tfrac{\lambda p}{s^*(\lambda)\beta_{22}^*(\nu_{22})} + M], \\
E(N_2, \xi = 0) = &\ \tfrac{\bar{b}_0}{\beta_0^*(\lambda)}[\tfrac{\lambda p}{s^*(\lambda)\beta_{22}^*(\nu_{22})}(1+\tfrac{\lambda}{\alpha}) + M], \\
E(N_2, \xi = id) = &\ \tfrac{\lambda p}{\alpha s^*(\lambda)\beta_{22}^*(\nu_{22})},
\end{aligned}$$

where

$$\begin{aligned}
D = &\ 2[\dot{\beta}_{11}^*(\nu_{11}) + (1-\beta_{11}^*(\nu_{11}))(\bar{r}_1 + \tfrac{1}{\nu_{11}})] + \lambda[(1-\beta_{11}^*(\nu_{11}))\bar{r}_1^{(2)} \\
&+2(\bar{r}_1 + \tfrac{1}{\nu_{11}})(\dot{\beta}_{11}^*(\nu_{11}) + \tfrac{1-\beta_{11}^*(\nu_{11})}{\nu_{11}})] + \tfrac{\lambda(1-p)\beta_{11}^*(\nu_{11})}{\beta_{12}^*(\nu_{12})}\{(1-\beta_{12}^*(\nu_{12}))\bar{r}_2^{(2)} \\
&+2(\bar{r}_2 + \tfrac{1}{\nu_{12}})(\dot{\beta}_{12}^*(\nu_{12}) + \tfrac{1-\beta_{12}^*(\nu_{12})}{\nu_{12}}) + \tfrac{2(1-\beta_{12}^*(\nu_{12}))(\bar{r}_2+\frac{1}{\nu_{12}})}{\beta_{12}^*(\nu_{12})}[\dot{\beta}_{12}^*(\nu_{12}) \\
&+(1-\beta_{12}^*(\nu_{12}))(\bar{r}_2 + \tfrac{1}{\nu_{12}})]\} - \tfrac{2\lambda(1-p)(1-\beta_{12}^*(\nu_{12}))\dot{\beta}_{11}^*(\nu_{11})}{\beta_{12}^*(\nu_{12})}(\bar{r}_2 + \tfrac{1}{\nu_{12}}),
\end{aligned}$$

$$\begin{aligned}
\bar{D} = &\ \tfrac{\lambda}{s^*(\lambda)\beta_{22}^*(\nu_{22})}\{\tfrac{\bar{b}_0^{(2)}}{\beta_0^*(\lambda)}[p\beta_{11}^*(\nu_{11}) + \tfrac{1-\rho}{1+\frac{\lambda \bar{b}_0}{\beta_0^*(\lambda)}}] + p\beta_{11}^*(\nu_{11})s^*(\lambda) \\
&\times[(1-\beta_{22}^*(\nu_{22}))\bar{r}_2^{(2)} + 2(\bar{r}_2 + \tfrac{1}{\nu_{22}})(\dot{\beta}_{22}^*(\nu_{22}) + \tfrac{1-\beta_{22}^*(\nu_{22})}{\nu_{22}})]\},
\end{aligned}$$

and

$$\begin{aligned}
L = &\ \tfrac{\lambda^2 p s^*(\lambda)\beta_{22}^*(\nu_{22})(D+\bar{D})}{2(1-\rho_1)(1-\rho)} + \tfrac{\lambda}{1-\rho}\{\tfrac{\lambda p}{\alpha}(1+\tfrac{\lambda \bar{b}_0}{\beta_0^*(\lambda)}) + \tfrac{1}{\beta_{11}^*(\nu_{11})}[\rho_0 + p\beta_{11}^*(\nu_{11}) \\
&\times[1 - s^*(\lambda) + \lambda s^*(\lambda)(\dot{\beta}_{22}^*(\nu_{22}) + (1-\beta_{22}^*(\nu_{22}))(\bar{r}_2 + \tfrac{1}{\nu_{22}}))] \\
&-\lambda p\dot{\beta}_{11}^*(\nu_{11})s^*(\lambda)\beta_{22}^*(\nu_{22})]\},
\end{aligned}$$

$$\begin{aligned}
M = &\ \tfrac{\lambda^2 p^2 \beta_{11}^*(\nu_{11})(D+\bar{D})}{2(1-\rho_1)(1-\rho)} + \tfrac{\lambda p}{1-\rho}\{\tfrac{\lambda p \beta_{11}^*(\nu_{11})}{\alpha s^*(\lambda)\beta_{22}^*(\nu_{22})}(1+\tfrac{\lambda \bar{b}_0}{\beta_0^*(\lambda)}) + \tfrac{1}{s^*(\lambda)\beta_{22}^*(\nu_{22})}[\rho_0 \\
&+p\beta_{11}^*(\nu_{11})[1 - s^*(\lambda) + \lambda s^*(\lambda)(\dot{\beta}_{22}^*(\nu_{22}) + (1-\beta_{22}^*(\nu_{22}))(\bar{r}_2 + \tfrac{1}{\nu_{22}}))]] \\
&-\lambda p\dot{\beta}_{11}^*(\nu_{11})\}.
\end{aligned}$$

7 Numerical Results

In this section we use the formulae derived previously to obtain numerical results and to investigate the way the mean number of customers in the retrial box $E(N_2)$ is affected when we vary the values of the parameters.

To construct the tables we assumed that the vacation time U_0, the service times, the start-up time and the repair times, follow exponential distributions with p.d.f.'s respectively,

$$\begin{array}{ll} b_0(x) = \frac{1}{\bar{b}_0} e^{-(1/\bar{b}_0)x}, & b_{ij}(x) = \frac{1}{\bar{b}_{ij}} e^{-(1/\bar{b}_{ij})x},\ i,j = 1,2, \\ s(x) = \frac{1}{\bar{s}} e^{-(1/\bar{s})x}, & r_j(x) = \frac{1}{\bar{r}_j} e^{-(1/\bar{r}_j)x},\ j = 1,2. \end{array}$$

Moreover we assume that in all tables below $\bar{b}_{12} = 0.33$, $\bar{r}_1 = 0.25$, $p = 0.5$, $\nu_{11} = 3$, $\nu_{12} = 4$.

Table 1 shows the way $E(N_2)$ changes when we vary the mean vacation time $\bar{b}_0$ for increasing values of the mean arrival rate λ. Here one can observe that even for a small value of λ, $\lambda = 0.2$ for example, $E(N_2)$ increases dramatically from 0.5895 to 796.81 when we pass from a system without vacation period ($\bar{b}_0 = 0$) to the system with $\bar{b}_0 = 2$. Moreover when the arrival rate λ increases to $\lambda = 0.5$, even a small change from $\bar{b}_0 = 0$ to $\bar{b}_0 = 0.2$ increases the mean number of retrial customers from 10.983 to 291.55. Thus we must be very careful on the vacation period that we must allow, in order to avoid a rather overloaded retrial box.

$\lambda \backslash \bar{b}_0$	0	0.2	0.33	0.5	1	2
0.2	0.5895	0.7528	0.8871	1.0935	2.1781	796.81
0.3	1.361	1.9342	2.5174	3.6847	75.904	
0.4	3.2738	5.9673	10.894	74.195		
0.45	5.553	14.307	104.85			
0.5	10.983	291.55				
0.55	34.323					

Table 1 : *Values of* $E(N_2)$ *for* $\bar{b}_{11} = 0.5$, $\bar{b}_{22} = 0.25$, $\bar{s} = 0.2$, $\bar{r}_2 = 0.25$, $\alpha = 0.8$, $\nu_{22} = 5$.

Table 2 contains values of $E(N_2)$ when we vary the mean retrial rate $E(\text{retrial}) = 1/\alpha$. The first column ($E(\text{retrial}) = 0$) corresponds to the $E(N_2)$ of our model assuming that $\alpha \to \infty$. One can observe here the increase of the mean number of retrial customers, an increase that is more apparent when λ increases. Moreover one can make conclusions on the mean retrial interval that must be allowed, in order to achieve a suitably small size of the retrial box.

$\lambda \backslash E(retrial)$	0	0.02	0.2	1	2	10
0.2	0.252	0.26	0.3322	0.6526	1.0532	4.2581
0.3	0.7735	0.7921	0.9593	1.7021	2.6306	10.059
0.4	2.8124	2.8628	3.3171	5.3363	7.8603	28.052
0.45	7.3029	7.415	8.4235	12.906	18.509	63.335
0.5	161.15	163.23	182.01	265.47	369.79	1204.4

Table 2 : $Values\ of\ E(N_2)\ for\ \bar{b}_0 = 0.2,\ b_{11} = 0.5,$
$\bar{b}_{22} = 0.25,\ \bar{s} = 0.2,\ \bar{r}_2 = 0.25, \nu_{22} = 5.$

$\lambda \backslash \bar{s}$	0.2	0.37	0.6	1.3	2.78
0.2	0.7528	0.868	1.0616	2.1158	354.24
0.3	1.9342	2.5135	3.7609	97.974	
0.4	5.9673	11.164	119.95		
0.45	14.307	128.85			
0.5	291.55				

Table 3 : $Values\ of\ E(N_2)\ for\ \bar{b}_{11} = 0.5,\ \bar{b}_{22} = 0.25,$
$\bar{b}_0 = 0.2,\ \bar{r}_2 = 0.25,\ \alpha = 0.8,\ \nu_{22} = 5.$

Table 3 contains values of $E(N_2)$ when we vary the mean start-up time $\bar{s}$, for increasing values of λ. We have to observe here the crusial role of start-up time in the evolution of our system. Note that for big values of this time period ($\bar{s} = 2.78$), even a small value of λ, $\lambda = 0.2$ for example, increase $E(N_2)$ to 354.24. This is quite reasonable if we realise that an arrival of a P_1 customer during the start-up period forces, the P_2 customer, again into the retrial box.

Table 4 shows the way $E(N_2)$ changes when we vary the mean repair time in second phase $\bar{r}_2$. One can observe again that the repair time plays an important role in the system performance as it increases in some cases dramatically, the mean number of retrial customers.

$\lambda \backslash \bar{r}_2$	0.1	0.25	0.4	0.65	1.3	2.6
0.2	0.6631	0.7528	0.8582	1.0811	2.2152	1279.6
0.3	1.5661	1.9342	2.4495	3.9389	578.09	
0.4	3.9381	5.9673	10.623	693.87		
0.45	7.0491	14.307	84.692			
0.5	16.16	291.55				
0.55	183.55					

Table 4 : $Values\ of\ E(N_2)\ for\ \bar{b}_{11} = 0.5,\ \bar{b}_{22} = 0.25,$
$\bar{b}_0 = 0.2,\ \bar{s} = 0.2,\ \alpha = 0.8,\ \nu_{22} = 5.$

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